



Impulse Responses by Local Projections: Practical Issues

by

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Motivation

- Impulse Responses are important statistics that substantiate models of the economy
- Is the DGP a VAR? Very likely it is not:
 - Zellner and Palm (1974) and Wallis (1977): a subset from a VAR follows a VARMA
 - Cooley and Dwyer (1998): many standard RBC models follow a VARMA
 - New solution techniques for nonlinear DSGE models produce polynomial difference equations

Disadvantages of VARs

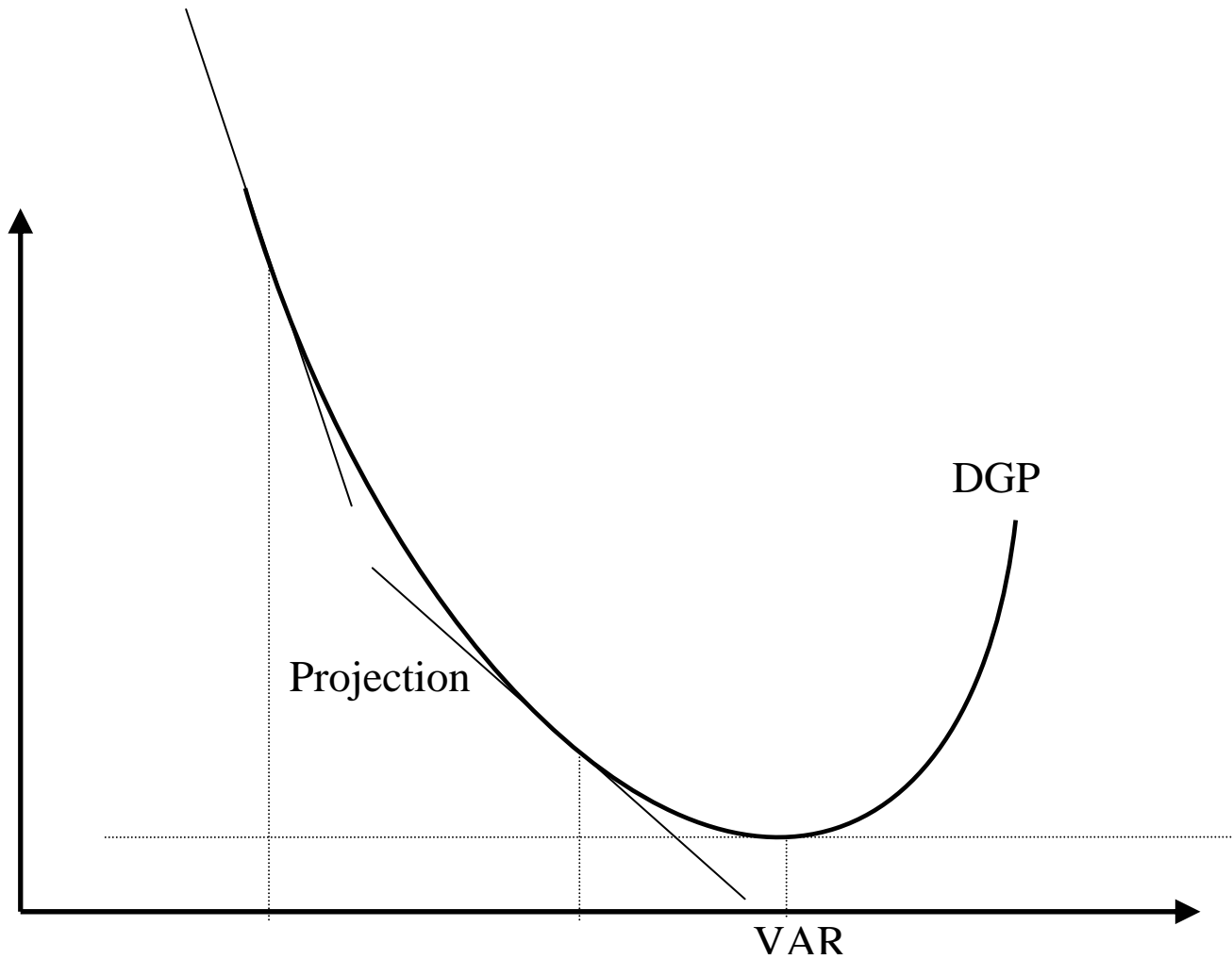
- VARs approximate the data *globally*: best, linear, one-step ahead predictors.
- Impulse responses are functions of multi-step forecasts
- Standard errors for impulse responses from VARs are complicated: they are highly nonlinear functions of estimated parameters

What this paper does

Because any model is a *global* approximation to the DGP in the sample, consider instead **local approximations for each forecast horizon** of interest

Use local projections!

Intuition



Advantages of Local Projections

- Can be estimated by **single-equation OLS** with standard regression packages
- Provide simple, analytic, **joint inference** for impulse response coefficients
- They are more **robust to misspecification**
- Experimentation with very **nonlinear and flexible** models is straight-forward

Estimation

A definition of impulse response (Hamilton, 1994, Koop et al. 1996):

$$IR(t, s, \mathbf{d}_i) = E(\mathbf{y}_{t+s} | \mathbf{v}_t = \mathbf{d}_i; X_t) - E(\mathbf{y}_{t+s} | \mathbf{v}_t = \mathbf{0}; X_t)$$

\mathbf{y}_t is $n \times 1$

$$X_t = (\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots)'$$

$$E(\mathbf{v}_t \mathbf{v}_t') = \Omega$$

$$D = [\mathbf{d}_1 \dots \mathbf{d}_i \dots \mathbf{d}_n];$$

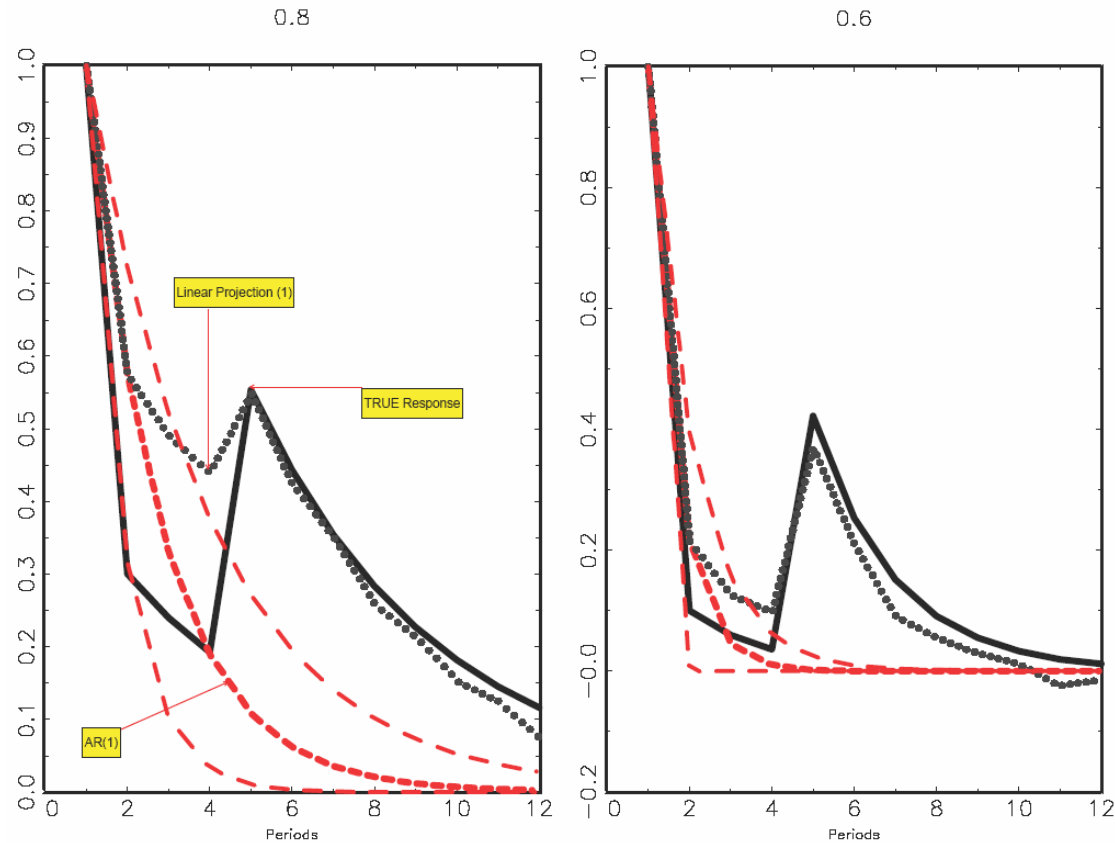
Hence, consider

$$\mathbf{y}_{t+s} = \alpha^s + B_1^{s+1} \mathbf{y}_{t-1} + \dots + B_p^{s+1} \mathbf{y}_{t-p} + \mathbf{u}_{t+s}^s$$

so that

$$IR(t, s, \mathbf{d}_i) = \hat{B}_1^s \mathbf{d}_i \quad s = 0, 1, 2, \dots, h$$

Example: AR(1) vs. Local Projection



$$y_t = \rho y_{t-1} + \varepsilon_t - 0.5\varepsilon_{t-1} + 0.4\varepsilon_{t-4}; T = 180; p = 1, 100 \text{ reps}$$

Practical Comments

- The maximum lag p need not be common to all s projections (e.g. VMA(q))
- The lag length and the IR horizon impose degree-of-freedom constraints for very small samples
- Consistency does not require that all $n \times h$ regressions be estimated jointly. It can be done by univariate regression for each n , and h .

Variance Decompositions

$$\mathbf{y}_{t+s} - E(\mathbf{y}_{t+s} | X_t) = \mathbf{u}_{t+s}^s \quad s = 0, 1, \dots, h$$

$$MSE_u(E(\mathbf{y}_{t+s} | X_t)) = E(\mathbf{u}_{t+s}^s \mathbf{u}_{t+s}^{s'})$$

$$MSE(E(\mathbf{y}_{t+s} | X_t)) = D^{-1} E(\mathbf{u}_{t+s}^s \mathbf{u}_{t+s}^{s'}) D'^{-1}$$

Structural Identification for Linear Projections

Example: Cholesky Decomposition

Estimate a VAR (also, the first projection):

$$\mathbf{y}_t = \alpha^0 + B_1^1 \mathbf{y}_{t-1} + \dots + B_p^1 \mathbf{y}_{t-p} + \mathbf{u}_t^0$$

$$E\mathbf{u}'\mathbf{u} = \Omega = A' \Lambda A$$

Then $D = A^{-1}$. This is the D for all subsequent projections.

Alternatively:

When available instruments can be used to resolve endogeneity, structural impulse responses can be calculated directly:

$$\mathbf{y}_{t+s} = \alpha^s + A_0^{s+1} \mathbf{y}_t + A_1^{s+1} \mathbf{y}_{t-1} + \dots + A_p^{s+1} \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_{t+s}^s$$

so that the response to the i^{th} variable is simply

$$IR(t, s, i) = \hat{A}(\cdot, i)_1^{s+1} \quad s = 0, 1, 2, \dots, h$$

Inference and Relation to VARs

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\Pi}' X_t + \mathbf{v}_t \quad \text{VAR}(p)$$
$$W_t \equiv \begin{bmatrix} \mathbf{y}_t - \boldsymbol{\mu} \\ \vdots \\ \mathbf{y}_{t-p+1} - \boldsymbol{\mu} \end{bmatrix}; F \equiv \begin{bmatrix} \boldsymbol{\Pi}_1 & \cdots & \boldsymbol{\Pi}_{p-1} & \boldsymbol{\Pi}_p \\ I & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}$$
$$\boldsymbol{\nu} = \begin{bmatrix} \mathbf{v}_t \\ \vdots \\ 0 \end{bmatrix}; \text{hence } W_t = FW_{t-1} + \boldsymbol{\nu}_t \quad \text{VAR}(1)$$

From the VAR(1) representation of the VAR(p),

$$\mathbf{y}_{t+s} - \boldsymbol{\mu} = \mathbf{v}_{t+s} + F_1^1 \mathbf{v}_{t+s-1} + \dots + F_1^s \mathbf{v}_t + F_1^{s+1} (\mathbf{y}_{t-1} - \boldsymbol{\mu}) + \dots + F_p^{s+1} (\mathbf{y}_{t-p} - \boldsymbol{\mu})$$

hence, as $s \rightarrow \infty$

$$\mathbf{y}_t = \boldsymbol{\gamma} + \mathbf{v}_t + F_1^1 \mathbf{v}_{t-1} + \dots + F_1^s \mathbf{v}_{t-s} + \dots$$

and therefore

$$\boxed{IR(t, s, \mathbf{d}_i) = F_1^s \mathbf{d}_i}$$

IR coefficients from a VAR(p)

$$F_1^1 = \Pi_1$$

$$F_1^2 = \Pi_1 F_1^1 + \Pi_2$$

⋮

$$F_1^s = \Pi_1 F_1^{s-1} + \Pi_2 F_1^{s-2} + \dots + \Pi_p F_1^{s-p}$$

Compare this with the linear projection

$$\mathbf{y}_{t+s} = \alpha^s + B_1^{s+1} \mathbf{y}_{t-1} + \dots + B_p^{s+1} \mathbf{y}_{t-p} + \mathbf{u}_{t+s}^s$$

then

$$\alpha^s = (I - F_1^s - \dots - F_p^s)$$

$$B_1^{s+1} = F_1^{s+1}$$

$$\mathbf{u}_{t+s}^s = \left(\mathbf{v}_{t+s} + F_1^1 \mathbf{v}_{t+s-1} + \dots + F_1^s \mathbf{v}_t \right)$$

Define $Y_t \equiv (\mathbf{y}_{t+1}, \dots, \mathbf{y}_{t+h})'$; $V_t \equiv (\mathbf{v}_{t+1}, \dots, \mathbf{v}_{t+h})'$; X_t

Under the assumption that the DGP is a VAR(p), consider the system:

$$Y_t = X_t \Psi + V_t \Phi$$

with

$$\Psi = \begin{bmatrix} F_1^1 & \dots & F_1^h \\ \vdots & \vdots & \vdots \\ F_p^1 & \dots & F_p^h \end{bmatrix}; \Phi = \begin{bmatrix} I_n & \dots & F_1^h \\ \vdots & \vdots & \vdots \\ 0 & \dots & I_n \end{bmatrix}$$

Further define:

$$E(\mathbf{v}_t \mathbf{v}_t') = \Omega_v, \text{ then } E(V_t V_t') = \Phi(I_h \otimes \Omega_v)\Phi' \equiv \Sigma$$

then

$$\text{vec}(\hat{\Psi}) = \left[(I \otimes X)' \Sigma^{-1} (I \otimes X) \right]^{-1} (I \otimes X)' \Sigma^{-1} \text{vec}(Y)$$

The usual impulse responses and their correct standard errors are rows 1- n and columns 1- nh of $\hat{\Psi}$

What does all this math mean?

- It establishes the **equivalence** between the IR coefficients from a VAR and from local projections
- It shows how to impose the VAR constraints to jointly estimate the local projections by **block-GLS**
- The GLS estimates deliver **efficient analytic inference** for IR coefficients **through time and across responses**.

Other remarks

- Monte Carlos show little loss of efficiency in estimating *univariate* local projections and using HAC robust standard errors (such as Newey-West)
- Denote $\hat{\Sigma}_L$ the HAC, VCV matrix of \hat{B}_1^s in the linear projection, then a 95% CI is $1.96 \pm (\mathbf{d}_i' \hat{\Sigma}_L \mathbf{d}_i)$
- Also, could use the $s-1$ stage residuals as regressors in the s stage projection
- Linear projections are a type of general misspecification test!

The Constrains of Linearity

- *Symmetry*
- *Shape invariance*
- *History independence*
- *Multidimensionality*

Flexible Local Projections

Generally,

$$\mathbf{y}_t = \Phi(\mathbf{v}_t, \mathbf{v}_{t-1}, \dots)$$

A Taylor series approximation to Φ is the Volterra series expansion (the non-linear Wold):

$$\begin{aligned} \mathbf{y}_t = & \sum_{i=0}^{\infty} \Phi_i \mathbf{v}_{t-i} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Phi_{ij} \mathbf{v}_{t-i} \mathbf{v}_{t-j} + \\ & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Phi_{ijk} \mathbf{v}_{t-i} \mathbf{v}_{t-j} \mathbf{v}_{t-k} + \dots \end{aligned}$$

It is natural to use polynomials for the local projections as well, for example:

$$\mathbf{y}_{t+s} = \alpha^s + B_1^{s+1} \mathbf{y}_{t-1} + Q_1^{s+1} \mathbf{y}_{t-1}^2 + C_1^{s+1} \mathbf{y}_{t-1}^3 + \\ B_2^{s+1} \mathbf{y}_{t-2} + \dots + B_p^{s+1} \mathbf{y}_{t-p} + \mathbf{u}_{t+s}^s$$

Hence

$$IR(t, s, \mathbf{d}_i) = \left\{ \begin{array}{l} \hat{B}_1^s \mathbf{d}_i + \hat{Q}_1^s (2\mathbf{y}_{t-1} \mathbf{d}_i + \mathbf{d}_i^2) + \\ \hat{C}_1^s (3\mathbf{y}_{t-1}^2 \mathbf{d}_i + 3\mathbf{y}_{t-1} \mathbf{d}_i^2 + \mathbf{d}_i^3) \end{array} \right\}$$

Remarks

- IRs no longer symmetric nor shape invariant
- IRs no longer history independent – they depend on \mathbf{y}_{t-1} . Evaluate at $\mathbf{y}_{t-1} = \bar{\mathbf{y}}$ to evaluate at linearity.
- Define $\lambda_i \equiv (\mathbf{d}_i \ 2\mathbf{y}_{t-1}\mathbf{d}_i + \mathbf{d}_i^2 \ 3\mathbf{y}_{t-1}^2\mathbf{d}_i + 3\mathbf{y}_{t-1}\mathbf{d}_i^2 + \mathbf{d}_i^3)'$ then a 95% CI is approximately

$$1.96 \pm (\lambda_i' \hat{\Sigma}_C \lambda_i)$$

Flexible projections in general

Since what matters are the terms associated with \mathbf{y}_{t-1} (but not the remaining lags), then

$$\mathbf{y}_{t+s} = m^s(\mathbf{y}_{t-1}; X_{t-1}) + \mathbf{u}_{t+s}^s$$

Thus, any parametric, semi-parametric and non-parametric conditional mean estimator will do.

Notice this can be done *univariately*.

Monte Carlo Simulations

Two experiments:

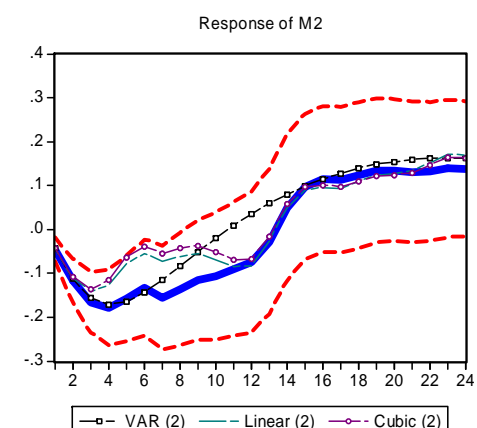
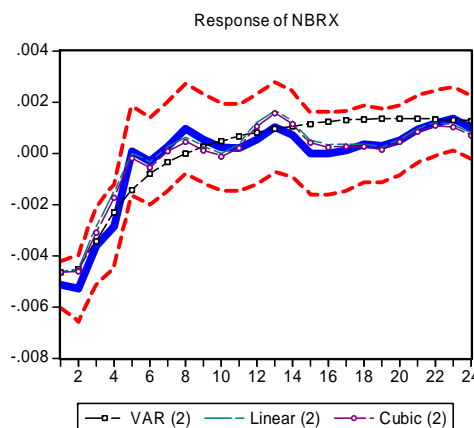
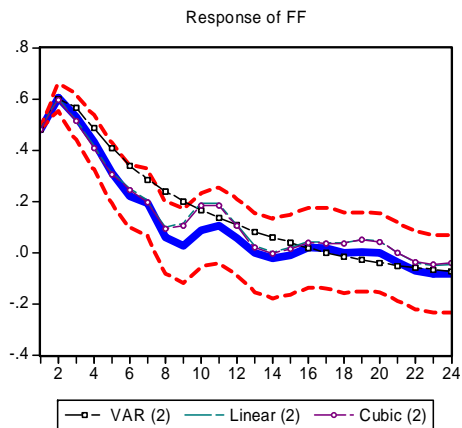
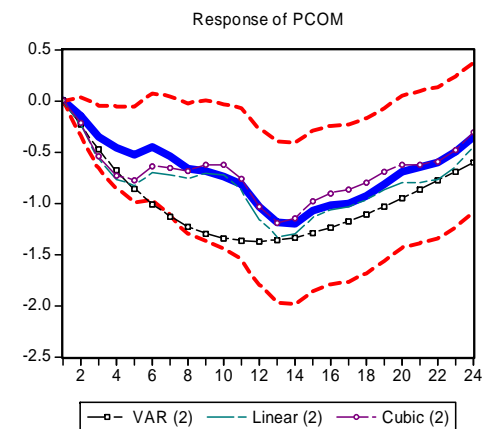
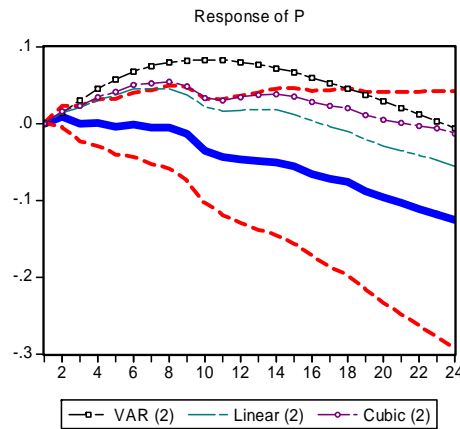
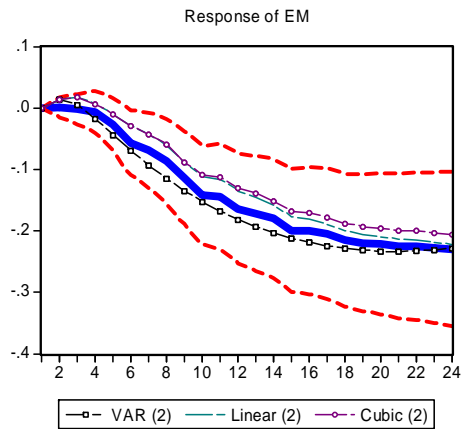
1. Robustness to lag-length misspecification, consistency and efficiency: Christiano, Eichenbaum and Evans (1996)
2. Robustness to nonlinearities: Jordà and Salyer (2003)

Robustness, Consistency and Efficiency

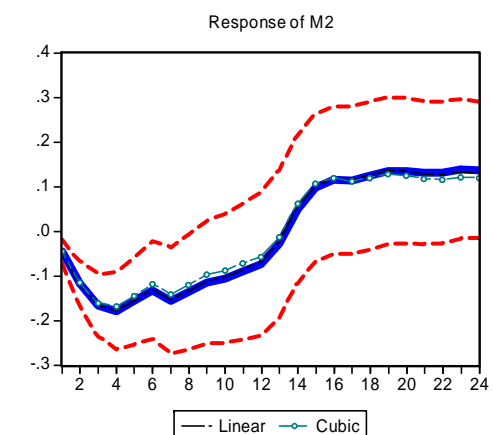
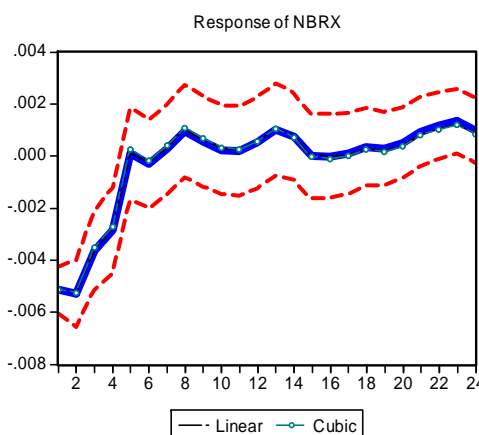
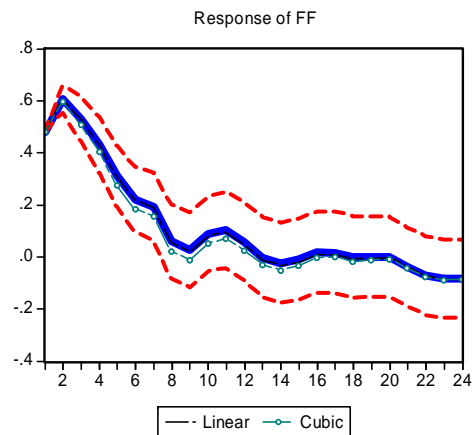
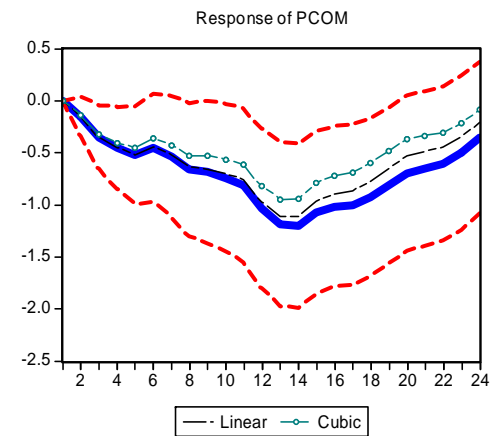
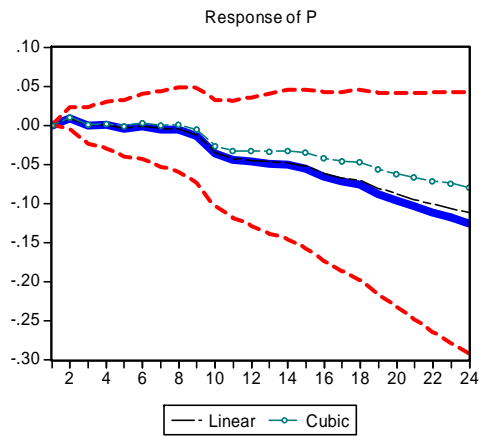
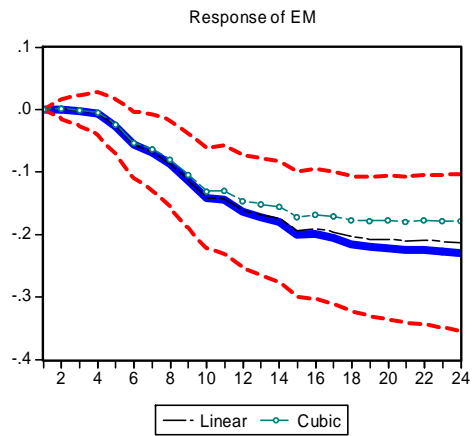
- Estimate the CEE VAR with 12 lags. Save the coefficients to produce the Monte Carlos

- Three experiments:
 1. Fit a VAR(2) and local-linear and –cubic projections: **Robustness**
 2. Fit local-linear and –cubic projections with 12 lags: **Consistency**
 3. Check standard errors from 2: **Efficiency**

Experiment 1



Experiment 2



Experiment 3

	EM			P			PCOM		
s	True-MC	Newey-West (Linear)	Newey-West (Cubic)	True-MC	Newey-West (Linear)	Newey-West (Cubic)	True-MC	Newey-West (Linear)	Newey-West (Cubic)
1	0.000	0.007	0.008	0.0000	0.007	0.007	0.000	0.089	0.096
...
12	0.046	0.044	0.048	0.042	0.042	0.045	0.390	0.380	0.416
...
24	0.064	0.063	0.068	0.086	0.081	0.086	0.371	0.431	0.484
	FF			NBRX			$\Delta M2$		
1	0.000	0.022	0.024	0.0005	0.0005	0.0005	0.014	0.012	0.014
...
12	0.077	0.075	0.083	0.0009	0.0009	0.0010	0.082	0.077	0.085
...
24	0.077	0.087	0.095	0.0006	0.0009	0.0010	0.078	0.088	0.096

Nonlinearities

Simulate:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} = A \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \\ y_{3t-1} \end{bmatrix} + Bh_{1t} + \begin{bmatrix} \sqrt{h_{1t}}\varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}; \quad \varepsilon_t \sim N(0, I)$$

$$h_{1t} = 0.5 + 0.3u_{t-1}^2 + 0.5h_{1,t-1}; \quad u_t = \sqrt{h_{1t}}\varepsilon_{1t}$$

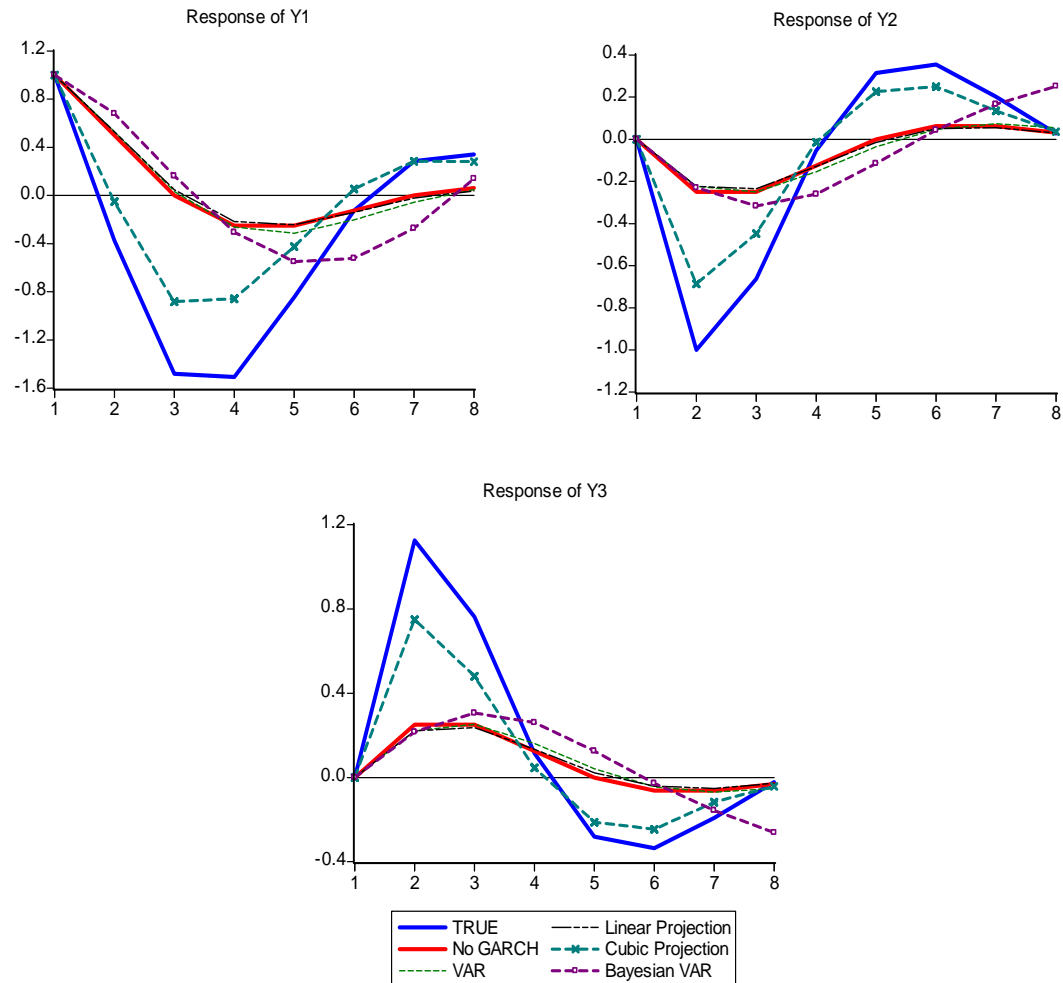
Compare:

- A VAR(1)
- Local-linear projections with one lag
- Local-cubic projections with one lag
- A Bayesian, time-varying parameter/volatility VAR à la Cogley and Sargent (2001,2003) - TVPVAR

The Monte-Carlo design for the TVPVAR:

- 100 obs. used to calibrate the prior
- Gibbs-sampler initialized with 2,000 draws
- Additional 5,000 draws to ensure convergence
- Select the quintiles of the distribution of the residuals of the first equation to identify 5 dates.
- Given the local histories of the 5 dates, calculate 100 Monte Carlo forecasts 1-8 steps ahead
- Obtain 5 impulse responses as the average of each 100 replications.
- Time of run: 9 days, 2 hours, 17 min. on a Sun Sunfire with 8, 900 Mhz processors and 16GB RAM

Nonlinearities



A New-Keynesian-Type Model of the Economy

Rudebusch and Svensson (1999) model:

- percentage gap between real GDP and potential GDP (from CBO)
- quarterly inflation in the GDP, chain-weighted price index in percent, annual rate
- quarterly average of the federal funds rate in percent at annual rate

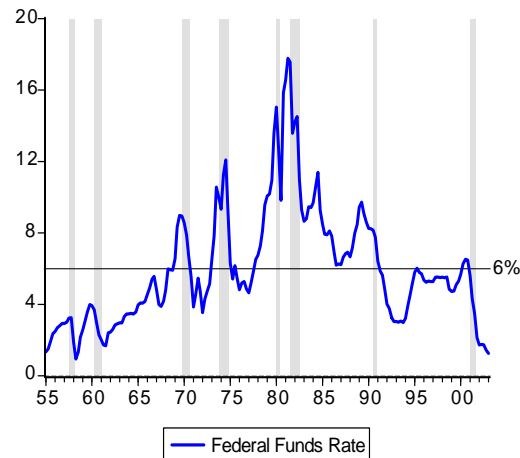
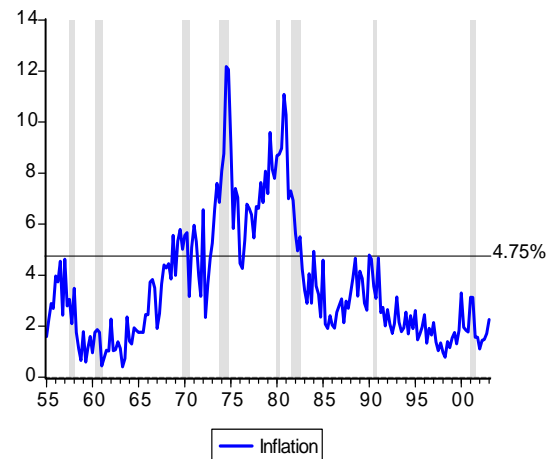
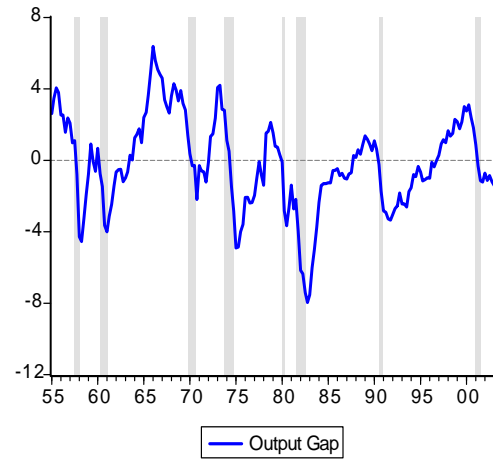
Asymmetries and Thresholds

Does the effectiveness of monetary policy depend on:

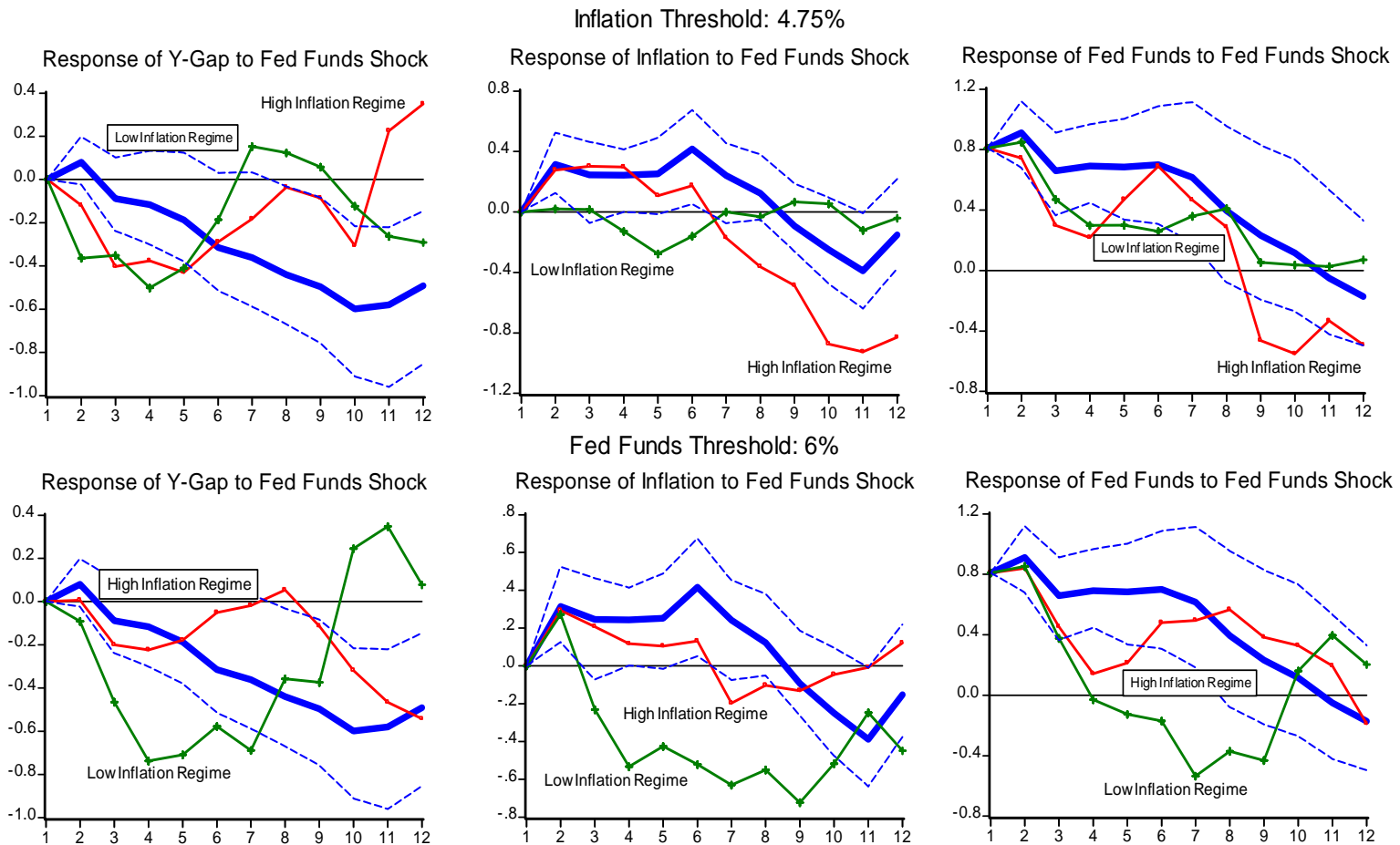
- The **stage of the business cycle**
- Whether **inflation is high or low**
- Whether **interest rates are close to the zero bound** or not.

I test for threshold effects with Hansen's (2000) test.

The Data



Thresholds in the New-Keynesian Model



Future Research

- Estimation of deep parameters in rational expectations models by efficient matching of MA coefficients.
- Applications to Panel Data and treatment effects.
- Efficiency improvements: using stage $s-1$ residuals as regressors in stage s projections
- Applications to non-Gaussian data

An Efficient Moment-Matching Estimator for RE models

Example (Fuhrer and Olivei, 2004):

$$z_t = \mu z_{t-1} + (\beta - \mu) E_t z_{t+1} + \gamma E_t x_t + \varepsilon_t$$

- z and x are the structural and driving processes
- ε is the stochastic shock
- For IS curve set z the output gap x the interest rate
- For AS curve set z inflation, x the output gap

The usual solution approach

Assume: $x_t = \alpha x_{t-1} + u_t$

Conjecture: $z_t = bz_{t-1} + cx_{t-1} + d\varepsilon_{t-1}$

Undetermined Coeffs.: $d = 1$

$$c = \frac{\gamma\alpha}{1 - (\beta - \mu)(b + \alpha)}$$

$$-(\beta - \mu)b^2 + b - \mu = 0$$

An alternative

Assume:
$$x_t = \sum_{i=0}^{\infty} \rho_i \varepsilon_{t-i} + \sum_{i=0}^{\infty} \theta_i u_{t-i}$$

Conjecture:
$$z_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i} + \sum_{i=0}^{\infty} b_i u_{t-i}$$

U.C.:

$$a_0 = (\beta - \gamma)a_1 + 1; \quad b_0 = (\beta - \mu)b_1$$

...

$$a_s = \mu a_{s-1} + (\beta - \mu)a_{s+1} + \gamma \rho_s$$

$$b_s = \mu b_{s-1} + (\beta - \mu)b_{s+1} + \gamma \theta_s$$

Define:

$$Y = (a_0 - 1, a_1, \dots, a_h, b_0, \dots, b_h)'$$

$$X_1 = (0, a_0, \dots, a_{h-1}, 0, b_0, \dots, b_{h-1})'$$

$$X_2 = (a_1, \dots, a_{h+1}, b_1, \dots, b_{h+1})'$$

$$X_3 = (0, \rho_1, \dots, \rho_{h-1}, 0, \theta_1, \dots, \theta_{h-1})'$$

Notice that the reduced form coefficients and the structural coefficients are related by:

$$Y - (X_1\mu + X_2(\beta - \mu) + X_3\gamma) = \omega$$

Let the covariance matrix of the impulse responses be:

$$\text{VAR}(Y) = \Omega_Y$$

Then, an efficient estimator for β, μ , and γ is:

$$\min \omega' \Omega_Y \omega$$

This also gives a natural metric for model fit even for models that would be rejected by FIML.

Identification with Long-Run Restrictions

Blanchard and Quah

Let $\mathbf{y}_t = A(L)\boldsymbol{\varepsilon}_t = \sum_{i=0}^{\infty} A_i \boldsymbol{\varepsilon}_{t-i}$ with $E\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = I$.

B-Q impose zero coefficient restrictions on $A(1)$.

Using the reduced form $\mathbf{y}_t = C(L)\mathbf{u}_t = \sum_{i=1}^{\infty} C_i \mathbf{u}_{t-i}$
and $E\mathbf{u}'\mathbf{u} = \Sigma$ we can recover the matrix D .

How to estimate an approximation to $C(1)$ with linear projections?

Option 1: Add the estimated linear projection coefficients,

$$\hat{C}(1) \cong \sum_{s=1}^h B_1^s$$

for h “large.”

Option 2:

Define: $\mathbf{Y}_{t+h} = \sum_{s=0}^h \mathbf{y}_{t+s}$.

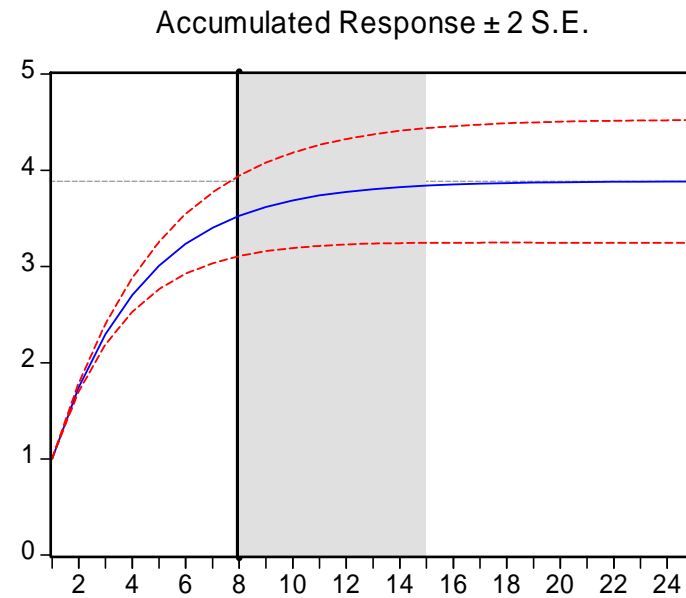
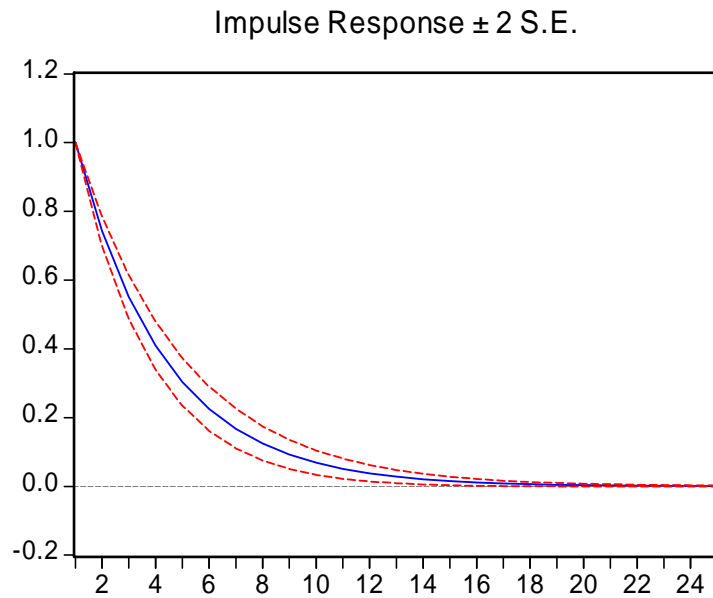
Then

$$\mathbf{Y}_{t+h} = G_1^h \mathbf{y}_{t-1} + \dots + G_p^h \mathbf{y}_{t-p} + \mathbf{u}_{t+h}^h$$

from which \hat{G}_1^h is an estimate of $\sum_{i=1}^h C_i$. Choose h “large.”

Example:

$$y_t = 0.75y_{t-1} + \varepsilon_t$$



A Panel Data Application

Example: Measuring the dynamic effect of a treatment.

“European Union Regional Policy and its Effects on Regional Growth and Labor Markets” by Florence Bouvet (Econ Dept, Graduate Student)

How does a poor region’s growth and unemployment respond to fund allocation from the European Regional Development Fund?

Data: panel of 111 regions from 8 EU countries, 1975-1999.

Instruments: political alignment between the regions, the national government and the EU Commission.

Estimation: TSLS with fixed country effects and fixed time effects.

Two examples: Response of Growth and Unemployment

Table 5: Explaining regional growth disparities

	OLS	OLS	2SLS	2SLS
Initial income per capita	-1.663*** (0.26)	-1.52*** (0.27)	-1.65* (0.9)	-1.21* (0.7)
ERDF per capita	-0.063*** (0.01)	-0.032** (0.01)	-0.173 (0.12)	-0.094 (0.08)
ERDF *_{poor}	0.084*** (0.03)			-0.109 (0.10)
Agriculture	-0.038*** (0.01)	-0.033*** (0.01)	0.024 (0.018)	0.01 (0.016)
Capital stock per worker	0.034 (0.03)	0.031 (0.04)	-0.051 (0.05)	-0.054 (0.04)
Adjusted R-square	0.185	0.183	0.188	0.204
Instruments			political alignment	political alignment
Partial R-square of the 1st stage of the IV estimation			0.232	0.272
Number of regions	98	98	75	75
Number of observations	2450	2450	1408	1408

$$\begin{aligned}
 Growth_{i,t} = & \beta_o + \beta_1 GVA_{i,1975} + \beta_2 ERDF_{i,t-1} + \beta_3 Agriculture_{i,t-1} \\
 & + \beta_4 Capital_{i,t-1} + D_c + D_t + \varepsilon_{i,t}
 \end{aligned}$$

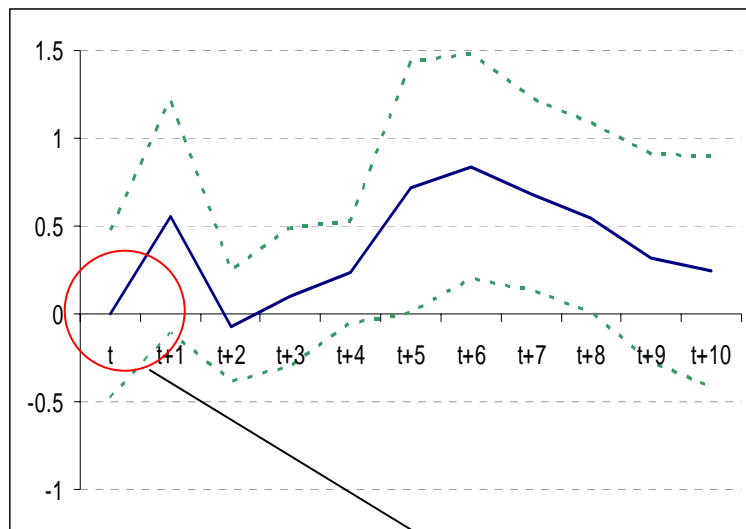
Table 9: Explaining regional unemployment disparities

	OLS	OLS	2SLS	2SLS	OLS	2SLS
Lagged unemployment					0.952*** (0.01)	1.00*** (0.02)
Growth rate of income per capita	0.151*** (0.04)	0.165*** (0.05)	0.019 (0.03)	-0.004 (0.03)	-0.025** (0.01)	-0.019* (0.01)
ERDF per capita	0.242*** (0.06)	0.324*** (0.06)	1.25*** (0.335)	0.173 (0.33)	0.300*** (0.008)	0.03 (0.06)
ERDF*poor	0.327*** (0.11)		0.249 (0.37)		-0.02* (0.01)	-0.384*** (0.14)
Agriculture	-0.013 (0.04)	-0.004 (0.04)	-0.118 (0.12)	0.222 (0.15)	0.012*** (0.005)	0.056*** (0.02)
Capital stock per worker	-0.253 (0.25)	-0.27 (0.25)	-0.157 (0.28)	-0.269 (0.31)	-0.032 (0.03)	-0.049 (0.03)
Adjusted R-square	0.540	0.529	0.157	0.532	0.935	0.929
Instruments			political alignment	political alignment		political alignment
R-square of the 1st stage of the IV estimation			0.241	0.180		0.262
number of regions	105	105	77	77	105	77
number of observations	2287	2287	1349	1349	2240	1343

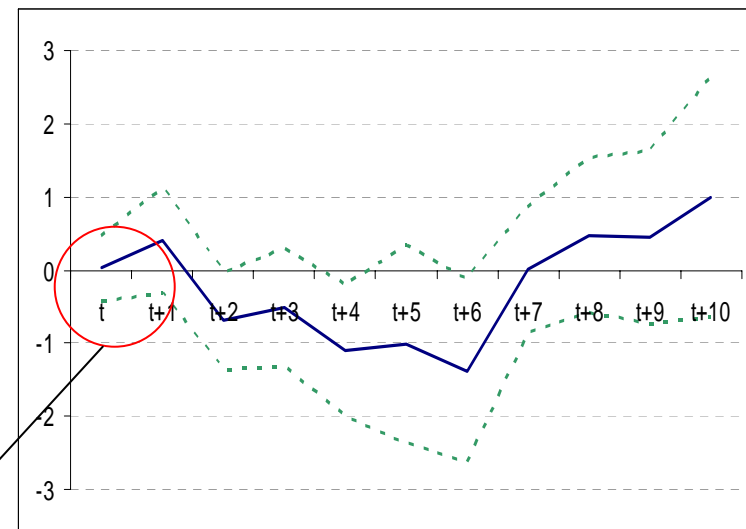
$$\begin{aligned}
 Unemployment_{i,t} = & \beta_0 + \beta_1 Growth_{i,t-1} + \beta_2 ERDF_{i,t-1} \\
 & + \beta_3 Agriculture_{i,t-1} + \beta_4 Capital_{i,t-1} + D_c + D_t + \varepsilon_{i,t}
 \end{aligned}$$

However,

Growth ($\Delta\%$)



Unemployment ($\Delta\%$)



These are the same coefficients as in the Tables!

Conclusions

- If the IR is the object of interest, concentrate on fitting the long-horizon forecasts rather than fitting the data one-period ahead.
- Projections can be estimated univariately – simplifies IR estimation and inference for panel/longitudinal data and non-Gaussian data.
- Consider graph analysis to resolve contemporaneous causality (Demiralp and Hoover)